

Tri-Coupled Bidirectional Duffing Oscillators: Applications of Fractional Calculus in Nonlinear Dynamics

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Resumen— En los últimos años, el Oscilador de Duffing ha sido ampliamente utilizado para modelar diversos fenómenos físicos y biológicos. Para el estudio de osciladores caóticos, el uso del cálculo fraccionario se ha vuelto indispensable, ya que la definición de Caputo permite modelar e implementar numéricamente fenómenos que no se ajustan completamente a derivadas de orden entero. En este trabajo, se integró el cálculo fraccionario a un Oscilador de Duffing triacoplado y bidireccional, lo que permitió analizar su dinámica bajo distintas condiciones. Se estudiaron sus respuestas temporales y se identificaron atractores caóticos al variar los parámetros de acoplamiento y el orden de las derivadas fraccionarias. Para resolver el modelo propuesto, se emplearon herramientas numéricas, y su dinámica se investigó mediante el análisis de diagramas de fase, series temporales y diagramas de bifurcación. Las conclusiones obtenidas arrojaron información relevante sobre el comportamiento del sistema bajo las diferentes condiciones evaluadas.

Palabras clave— Sistemas dinámicos, Calculo fraccionario, Oscilador Duffing, Acoplamiento bidireccional.

Abstract— In recent years, the Duffing Oscillator has been extensively applied to model a variety of physical and biological phenomena. The study of chaotic oscillators often necessitates the use of fractional calculus, as Caputo's definition enables the modeling and numerical implementation of systems that do not conform strictly to integer-order derivatives. In this work, fractional calculus was incorporated into a tri-coupled, bidirectional Duffing Oscillator to analyze its dynamics under varying conditions. The investigation focused on temporal responses and the identification of chaotic attractors as coupling parameters and fractional derivative orders were adjusted. Numerical methods were employed to solve the proposed model, and the dynamics were explored through phase plane analysis, time series, and bifurcation diagrams. The findings provide key insights into the behavior of the system across different parameter regimes.

Keywords— Dynamic system, Duffing oscillator, Fractional order, Bidirectional coupling.

I. INTRODUCTION

The Duffing Oscillator is a non-linear differential equation devised in 1918 by G. Duffing [1]. Its applications are notable in physics, biology, engineering, signal processing, and communication systems, among others [2-5]. Particularly, it is usually studied by coupling differential equations, creating complex systems to perform stability analysis, the analysis of its synchronization, and emerging chaos in it [6-8].

On the other hand, fractional calculus has been highlighted for its usefulness in physical applications [9]. Caputo's definition offers a way to perform fractional derivatives compatible with classical initial conditions [10]. This opens the door to model phenomena that do not fit traditional differential equations, such as Duffing Oscillators. By incorporating fractional derivatives into the equations of motion, we can capture the memory and non-locality inherent to these systems, allowing us to understand their complex behavior better [10-13]. The aim in this paper is studies a configuration of tri-coupled and bidirectional Duffing oscillators, analyzing both their temporal responses and their chaotic attractors under various coupling conditions. Fractional calculus and bifurcation concepts are integrated to model memory and dissipation effects that cannot be adequately described by traditional formulations.

The paper is organized as follows: in section II, the basic elements to obtain the tri-coupled and bidirectional Duffing oscillator model are addressed, using fractional calculus to find its solution. In section III, an analysis of the results obtained from the proposed model is performed, explaining the dynamics using phase diagrams, time series, and bifurcation diagrams. Finally, in section IV, conclusions are presented, and the most relevant results are discussed.

II. MODEL

Consider the usual form of the Duffing oscillator, without a coupling term, written as:

$$\frac{d^2 x}{dt^2} + \alpha \frac{dx}{dt} + \omega_0^2 x + \delta x^3 = 0, \quad (1)$$

where α , ω_0^2 , y δ are non-trivial constants. Caputo [10] defines the fractional derivative as:

$$\frac{d^\lambda f}{dt^\lambda} = \frac{1}{\Gamma(n-\lambda)} \int_0^t \frac{f^{(n)}(\eta)}{(t-\eta)^{\lambda-n+1}} d\eta, \quad (2)$$

where $n-1 < \lambda < n$, $\lambda \in \mathbb{R}$ corresponds to the order of the fractional derivate, η is a variable of integration and $f^{(n)}$ is an ordinary derivative. Then, extend the above equation to the fractional order given by:

$$\frac{d^2 x}{dt^2} + \alpha \frac{d^\lambda x}{dt^\lambda} + \omega_0^2 x + \delta x^3 = 0, \quad (3)$$

where $0 < \lambda < 2$ It is a real number and corresponds to the order of the fractional derivative. Thus, assuming that $x_1, x_2, x_3: [0, \infty) \rightarrow \mathbb{R}$, complete system is described by the following set of equations:

$$\begin{aligned} \frac{d^2 x}{dt^2} + \frac{d^\lambda x_1}{dt^\lambda} + \omega_0^2 x_1 + \delta x_1^3 + \sigma(x_1 - x_2) + \sigma(x_1 - x_3) &= 0 \\ \frac{d^2 x}{dt^2} + \frac{d^\lambda x_2}{dt^\lambda} + \omega_0^2 x_2 + \delta x_2^3 + \sigma(x_2 - x_1) + \sigma(x_2 - x_3) &= 0 \quad (4) \\ \frac{d^2 x}{dt^2} + \frac{d^\lambda x_3}{dt^\lambda} + \omega_0^2 x_3 + \delta x_3^3 + \sigma(x_3 - x_1) + \sigma(x_3 - x_2) &= 0 \end{aligned}$$

Where $\sigma > 0$ is the coupling order.

The figure 1 shows the system of three bidirectionally coupled Duffing oscillators using a directed graph, where the vertices represent each of the oscillators and the edges represent their coupling.

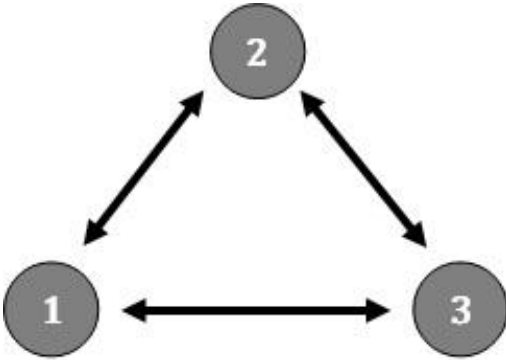


Fig. 1. Diagram of three bi-directionally coupled Duffing oscillators [14].

To solve the System of equations (4), it is convenient to rewrite them as follows:

$$\frac{d^\lambda x_m}{dt^\lambda} = y_m, \quad (5)$$

$\frac{d^\lambda y_m}{dt^{\lambda_1}} = -\alpha y_m - \omega_0^2 x_m - \delta x_m^3 - \sigma(x_m - x_n)$,
with $m, n = 1, 2, 3$. Here, the additive rule for the indices in the fractional derivatives is used as

$$\frac{d^{q_1}}{dt^{q_1}} \left(\frac{d^\lambda}{dt^\lambda} \right) = \frac{d^{q_1+\lambda}}{dt^{q_1+\lambda}}, \quad (6)$$

with the restriction $q_1 + \lambda = 2$.

It is possible to implement a numerical scheme to solve (5) based on equation (6) as follows [15-16].

$$x_m(t_k) = h^\lambda y_m(t_{k-1}) - \sum_{j=1}^k c_j^{(\lambda)} x_m(t_{k-j}), \quad (7)$$

$$\begin{aligned} y_m(t_k) = h^{q_1} &- \alpha y_m(t_{k-1}) - \omega_0^2 x_m(t_{k-1}) - \delta x_m^3(t_{k-1}) - \\ &\sigma[x_m(t_{k-1}) - x_{n_1}(t_{k-1})] - \sigma[x_m(t_{k-1}) - \\ &x_{n_2}(t_{k-1})] - \sum_{j=1}^k c_j^{(q_1)} y_m(t_{k-j}), \end{aligned} \quad (8)$$

where $t_k = kh$ for $k = 1, \dots, N$ with $h = \frac{T}{N}$ and T is the size of the time interval.

III. DYNAMIC SYSTEM RESULTS

To analyze the dynamics of the bidirectionally Tri-coupled Duffing Oscillator, the behavior of the solutions obtained using the algorithm proposed by Caputo (7) and (8) is shown, for some coupling strengths σ and fractional order derivatives λ that present interesting dynamics.

First, the time series and the phase plane of the system are shown in the figure 2, considering a coupling strength $\sigma = 0.1$ with fractional derivative $\lambda = 0.8$. It is observed that the time series for the state variables x_2 and x_3 tend to be a steady state with time, note that the state variable x_1 is identical to x_3 due to the symmetry of the coupling. These results can be confirmed in its phase plane where is observed the evolution of the dynamics of the system, noting that the system tends to an attractor point.

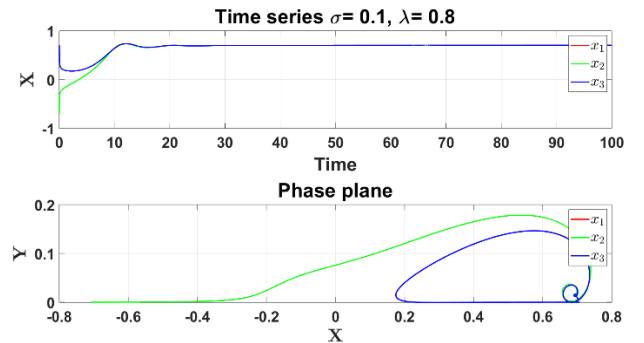


Fig. 2. Time series and phase diagram corresponding to $\sigma = 0.1$ and $\lambda = 0.8$.

Considering a coupling strength $\sigma = 0.1$ with fractional derivative $\lambda = 1.98$, the time series and the phase plane of the system are observed as shown in figure 3, where the time

series for the state variables x_2 and x_3 tend to a steady state, however, they present oscillations during the simulation time. These results can be confirmed in its phase plane where it is observed that the dynamics of the system tend to be a deformed limit cycle.

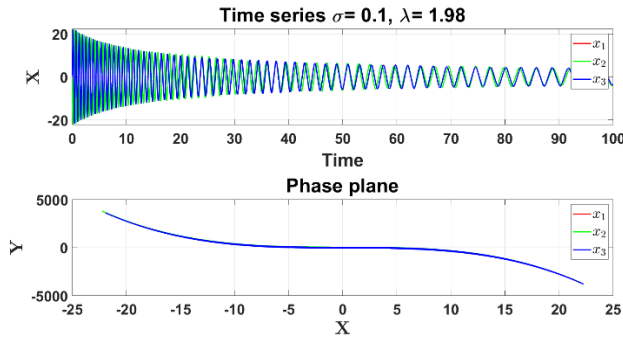


Fig. 3. Time series and phase diagram corresponding to $\sigma = 0.1$ and $\lambda = 1.98$.

The figure 4 exhibit the time series and the phase plane of the system considering a coupling strength $\sigma = 0.75$ with fractional derivative $\lambda = 0.8$. The time series for the state variables x_2 and x_3 quickly tend to be a stationary state with time. These results are confirmed in its phase plane where is observed the evolution of the dynamics of the system, which is chaotic at first and culminates by entering an attractor point.

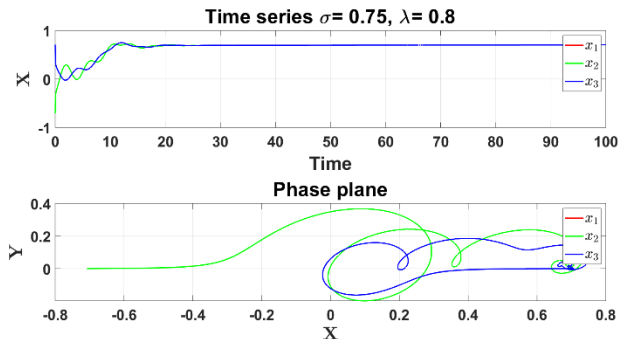


Fig. 4. Time series and phase diagram corresponding to $\sigma = 0.75$ and $\lambda = 0.8$.

Next, in the figure 5 is shown the time series and the phase plane of the system, considering a coupling strength $\sigma = 0.1$ with fractional derivative $\lambda = 0.8$. The time series for the state variables x_2 and x_3 end to an oscillatory steady state with time. These results are confirmed in its phase plane where is observed the evolution of the dynamics of the system, in which is appreciated a deformed limit cycle.

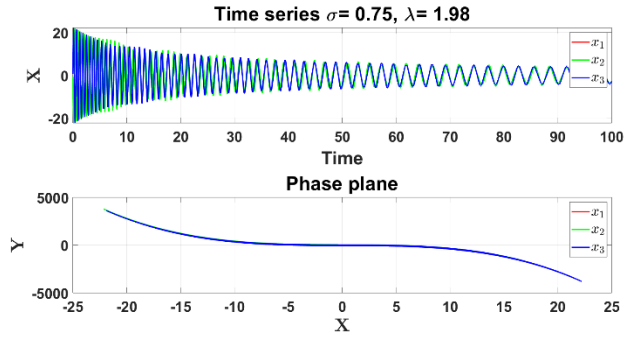


Fig. 5. Time series and phase diagram corresponding to $\sigma = 0.75$ and $\lambda = 1.98$.

To study the dynamics of the bidirectional Duffing Oscillator under different couplings the bifurcation diagrams are analyzed considering σ as bifurcation parameter, fixing the fractional derivative λ . Some cases with interesting dynamics are shown.

Setting the fractional derivative at $\lambda = .8$, exhibits a fairly stable and consistent behavior as shown in figure 6, when calculating its maximum.

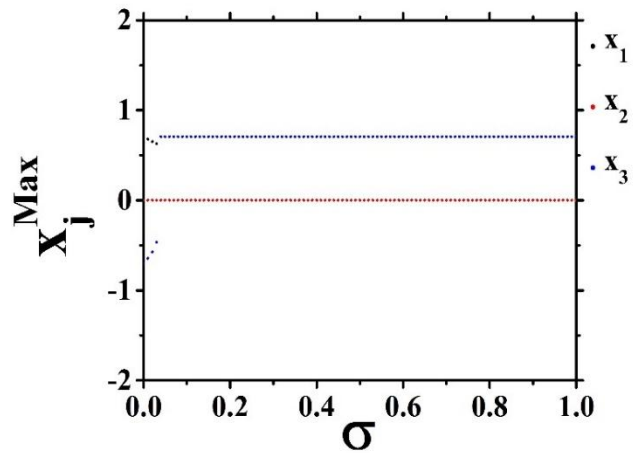


Fig. 6. Bifurcation diagram of the parameter σ using $\lambda = 0.8$.

In the figure 7 is shown the bifurcation diagram whit fractional derivative settled as $\lambda = 0.24$, it exhibits a fairly stable and consistent behavior when calculating its maxima, however, when approximating the coupling the solution begins to vary, and the solutions x_1 and x_3 acquire their own and different dynamics.

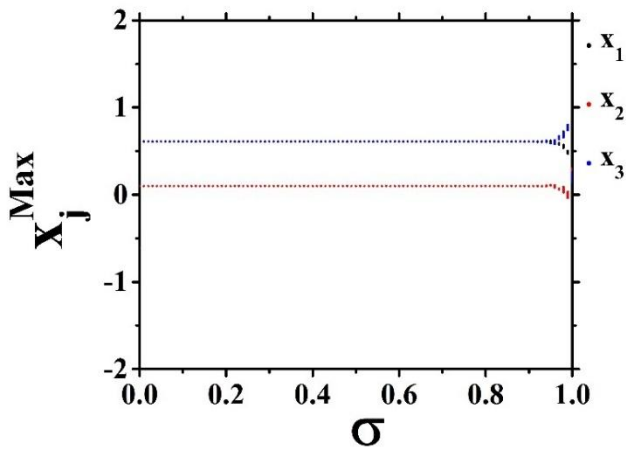


Fig. 7. Bifurcation diagram of the parameter σ using $\lambda = 0.24$.

It is important to note that approximating the fractional derivative to $\lambda = 2$ leads to chaotic behavior in the system, this phenomenon as it can be observed in figure 8 where the state variables, by increasing the coupling, is obtained a system with greater chaos, which can explain the oscillatory behavior in Figures 3 to 5.

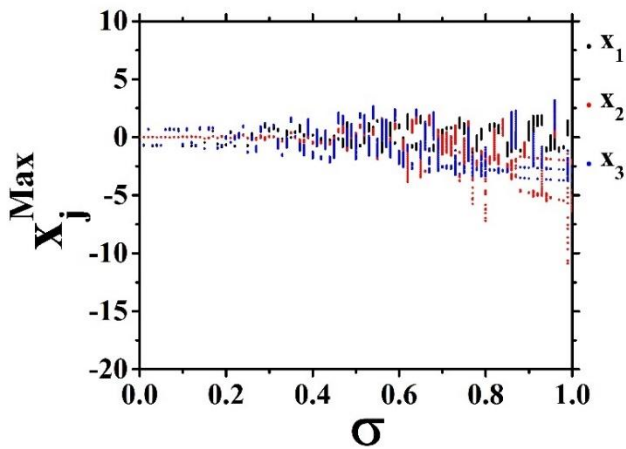


Fig. 8. Bifurcation diagram of the parameter σ using $\lambda = 1.98$.

IV. CONCLUSIONS

In response to the objective proposed in this work we have demonstrated how tri-coupled bidirectional Duffing oscillators, modeled through fractional calculus, provide a powerful tool for the study of complex phenomena in nonlinear dynamics. The application of fractional calculus has enabled a deeper analysis of the nonlinear interactions and long-memory effects exhibited by these systems. It has been observed that bidirectional coupling between oscillators introduces a rich variety of dynamic behaviors, observed in bifurcation diagrams, and phase transitions, which are not evident in linear systems or traditional models based on integer-order calculus [14].

Several areas deserve further investigation, such as image encryption or the search for invariants in one-way ring systems, as they could contribute significantly to knowledge.

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